# Microeconomics Tutorial I : Basic Topology

Course Teacher: Prof. Sushama Murty
TA: Noyel Sebastian
CSWE/JNU

August 13, 2025

# Contents

# Euclidean Space

- $n$-dimensional Euclidean space $\mathbb{R}^n$
  - Non-negative orthant of $\mathbb{R}^n$ - $\mathbb{R}^n_+$
  - Non-positive orthant of $\mathbb{R}^n$ - $\mathbb{R}^n_-$
  - Positive orthant of $\mathbb{R}^n$ - $\mathbb{R}^n_{++}$
  - Negative Orthant of $\mathbb{R}^n$ - $\mathbb{R}^n_{--}$
- Example: case $n = 1$: $\mathbb{R}^1 = \mathbb{R}$
  - $\mathbb{R}_+ = [0, \infty)$
  - $\mathbb{R}_- = (-\infty, 0]$
  - $\mathbb{R}_{++} = (0, \infty)$
  - $\mathbb{R}_{--} = (-\infty, 0)$
- Example : case $n = 2$: $\mathbb{R}^2$

# Euclidean Space & Neighbourhood

Euclidean Norm

- The length of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, denoted $\|\mathbf{x}\|$, and also called the *Euclidean norm* of $\mathbf{x}$, is given by the equation

$$\|\mathbf{x}\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

- Norm measures the length ie, the distance between the origin and a point.

# Euclidean Space & Neighbourhood

Euclidean Norm

- The length of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, denoted $\|\mathbf{x}\|$, and also called the *Euclidean norm* of $\mathbf{x}$, is given by the equation

$$\|\mathbf{x}\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

- Norm measures the length ie, the distance between the origin and a point.

Euclidean Distance

- The *Euclidean distance* between two vectors $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in $\mathbb{R}^n$ is the length of the vector $\tilde{\mathbf{x}} - \mathbf{x}$, i.e.,

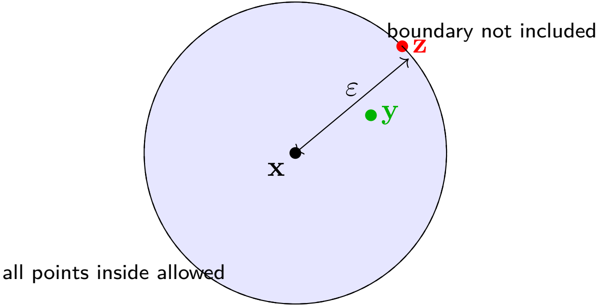
$$\|\tilde{\mathbf{x}} - \mathbf{x}\| = \sqrt{(\tilde{x}_1 - x_1)^2 + (\tilde{x}_2 - x_2)^2 + \cdots + (\tilde{x}_n - x_n)^2}.$$

# Epsilon Ball and Neighbourhood

Epsilon Ball

An $\varepsilon$-*ball* around a point $\mathbf{x}$ in a space (usually $\mathbb{R}^n$) is the set of all points within a distance $\varepsilon$ from $\mathbf{x}$, *but not including the boundary* if it's an open ball.

$$B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < \varepsilon\}$$



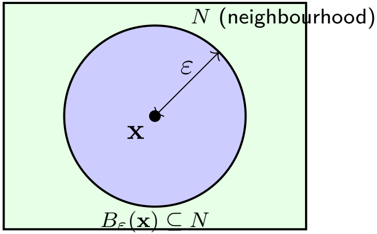
# Epsilon Ball and Neighbourhood

**Neighbourhood in $\mathbb{R}^n$**

A *neighbourhood* of a point $\mathbf{x}$ is **any set that contains an $\varepsilon$-ball around** $\mathbf{x}$ for some $\varepsilon > 0$.

That is:

$$N \text{ is a neighbourhood of } \mathbf{x}(N_\epsilon(x)) \iff \exists\, \varepsilon > 0 \text{ such that } B_\varepsilon(\mathbf{x}) \subseteq N$$



# Neighbourhood

**Neighbourhood in $\mathbb{R}$**

say, $x$ be any point on a number line.
A set $N$ is a neighbourhood of $x$ if there's some $\varepsilon > 0$ such that:

$$(x - \varepsilon,\ x + \varepsilon) \subseteq N.$$

Example in class

**Punctured Neighborhood around x**

$$N_\epsilon(x) \setminus \{x\}$$

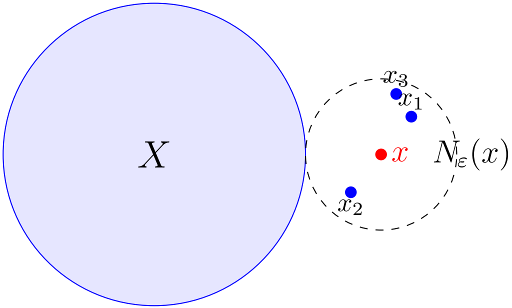
We deduct the point $x$ and measure the distance.

# Limit Point

**Limit Point:** A point $x \in \mathbb{R}^n$ is a **limit point** of a set $X \subset \mathbb{R}^n$ if every neighborhood around $x$ contains at least one point from $X$ that is not equal to $x$, i.e.,

$$\forall \varepsilon > 0, \quad (N_\varepsilon(x) \setminus \{x\}) \cap X \neq \emptyset.$$

That is, every punctured neighborhood of $x$ contains at least one point of $X$.



# Closed Set

**Definition:** A set $X \subseteq \mathbb{R}^n$ is **closed** if it contains all its limit points.

ie, If a sequence $\{x_n\} \subset X$ converges to a point $x$, then $x \in X$.

Example:

- Let $X = [0, 1] \subset \mathbb{R}$.
- Take any sequence $\{x_n\} \subset [0, 1]$ that converges to a point $x \in \mathbb{R}$.
- Since the endpoints $0$ and $1$ are included in the interval, any limit point of such a sequence will also lie within $X$.

Therefore, $X$ contains all its limit points, and hence, $X$ is a **closed set**.

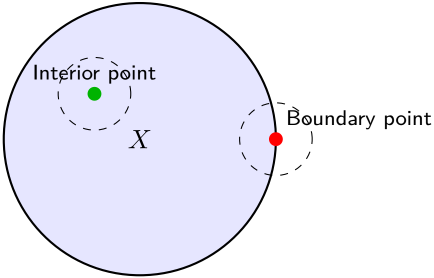
# Interior and Boundary Points

**Interior Point:** A point $x \in X$ is an interior point if there exists $\varepsilon > 0$ such that:

$$B_\varepsilon(x) \subseteq X$$

**Boundary Point:** A point $x \in \mathbb{R}^n$ is a boundary point of $X$ if:

$$\forall \varepsilon > 0, \quad B_\varepsilon(x) \cap X \neq \emptyset \quad \text{and} \quad B_\varepsilon(x) \cap (\mathbb{R}^n \setminus X) \neq \emptyset$$

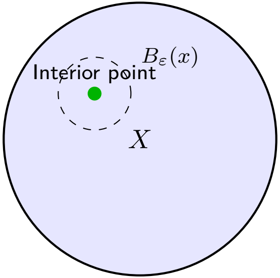


# Open Set

**Definition:** A set is open if for all $x \in X \quad \exists$ some $\varepsilon -$ neighbourhood of x, $N_\varepsilon(x)$, such that $N_\varepsilon(x) \subseteq X$

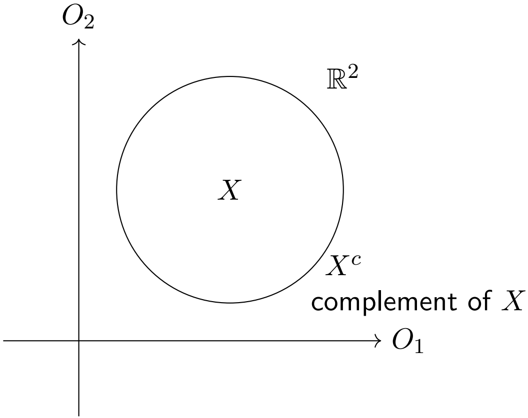
$$\forall x \in X, \quad \exists\ \varepsilon > 0 \text{ such that } N_\varepsilon(x) \subseteq X$$

ie, Every point in the set is an interior point. You can draw a small -ball around any point in $X$ that stays entirely within $X$.



# Some Results

- Complements of open sets are closed.
- Complements of closed sets are open.



# Problem 1

**Q:** Prove that the empty set $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

# Problem 1

**Q:** Prove that the empty set $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

**Solution:**

1. $\varnothing$ is open: By definition, every point of $\varnothing$ is an interior point (vacuous truth). So $\varnothing$ is open.

# Problem 1

**Q:** Prove that the empty set $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

**Solution:**

1. $\varnothing$ is open: By definition, every point of $\varnothing$ is an interior point (vacuous truth). So $\varnothing$ is open.
2. $\mathbb{R}^n$ is open: For any $x \in \mathbb{R}^n$, pick any $\varepsilon > 0$. Then $N_\varepsilon(x) \subset \mathbb{R}^n$. Thus every point is interior, so $\mathbb{R}^n$ is open.

# Problem 1

**Q:** Prove that the empty set $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

**Solution:**

1. $\varnothing$ is open: By definition, every point of $\varnothing$ is an interior point (vacuous truth). So $\varnothing$ is open.
2. $\mathbb{R}^n$ is open: For any $x \in \mathbb{R}^n$, pick any $\varepsilon > 0$. Then $N_\varepsilon(x) \subset \mathbb{R}^n$. Thus every point is interior, so $\mathbb{R}^n$ is open.
3. Closedness: A set is closed iff its complement is open. $\varnothing^c = \mathbb{R}^n$ is open, so $\varnothing$ is closed. Similarly, $(\mathbb{R}^n)^c = \varnothing$ is open, so $\mathbb{R}^n$ is closed.

# Problem 2

**Q:** Prove that the set of integers, $\mathbb{I}$, is a closed set in $\mathbb{R}$.

**Solution:**

- The complement of $\mathbb{I}$ in $\mathbb{R}$ is $\mathbb{R} \setminus \mathbb{I}$, which contains all real numbers that are not integers.
- Let $x \in \mathbb{R} \setminus \mathbb{I}$. Then $x$ lies between two consecutive integers $a$ and $a + 1$, i.e., $a < x < a + 1$.
- There exists $\varepsilon > 0$ such that the open interval $(x - \varepsilon, x + \varepsilon) \subset (a, a + 1)$.
- This interval contains no integers, so it lies entirely within $\mathbb{R} \setminus \mathbb{I}$.
- Hence, every point in $\mathbb{R} \setminus \mathbb{I}$ has a neighborhood contained in the complement.
- Therefore, $\mathbb{R} \setminus \mathbb{I}$ is open, so $\mathbb{I}$ is closed.

# Problem 3

**Q:**Prove that (A)a union of any number (finite or infinite) of open sets is open, and (B)an intersection of any number (finite or infinite) of closed sets is closed.

**A:** We need to show, say, $U_\alpha$ is a collection of open sets then $\bigcup_\alpha U_\alpha$ is also open.
Take an element $x \in \bigcup_\alpha U_\alpha$

# Problem 3

**Q:** Prove that (A)a union of any number (finite or infinite) of open sets is open, and (B)an intersection of any number (finite or infinite) of closed sets is closed.

**A:** We need to show, say, $U_\alpha$ is a collection of open sets then $\bigcup_\alpha U_\alpha$ is also open.
Take an element $x \in \bigcup_\alpha U_\alpha$
We need to show, $x \in U_k$ for some $k \Rightarrow x$ belongs to at least one of the open set in the union.

$$x \in U_k \quad \exists \quad N_\varepsilon(x) \subseteq U_k$$

# Problem 3

**Q:**Prove that (A)a union of any number (finite or infinite) of open sets is open, and (B)an intersection of any number (finite or infinite) of closed sets is closed.

**A:** We need to show, say, $U_\alpha$ is a collection of open sets then $\bigcup_\alpha U_\alpha$ is also open.
Take an element $x \in \bigcup_\alpha U_\alpha$
We need to show, $x \in U_k$ for some $k \Rightarrow x$ belongs to at least one of the open set in the union.

$$x \in U_k \quad \exists \quad N_\varepsilon(x) \subseteq U_k$$

but $U_k$ is also a subset of our union of open sets.

$$x \in U_k \quad \exists \quad N_\varepsilon(x) \subseteq U_k \subseteq \bigcup_\alpha U_\alpha$$

ie,

$$N_\varepsilon(x) \subseteq \bigcup_\alpha U_\alpha \quad \therefore \quad \bigcup_\alpha U_\alpha \quad \text{is open}$$

# Problem 3

**B** We need to show, if $U_\alpha$ is a collection of closed sets, then $\bigcap_\alpha U_\alpha$ is also a closed set.
De Morgan's Law:

$$(A \bigcup B)^c = A^c \bigcap B^c$$

$$(A \bigcap B)^c = A^c \bigcup B^c$$

From **A**, we showed that the arbitrary union of open sets is open.
Thus,

$$(\bigcup_\alpha U_\alpha^c)^c = \bigcap U_\alpha \quad \text{is closed}$$

# Problem 4

**Q:** Find $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$. What does this example demonstrate about open sets?

**Solution:**

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}$$

Since 0 is the only point contained in all intervals $\left(-\frac{1}{n},\frac{1}{n}\right)$, the intersection is the singleton set $\{0\}$.

**Insight:** Arbitrary intersections of open sets need not be open.

# Problem 5

**Q:** Find $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right]$. What does this example demonstrate about closed sets?

**Solution:**

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right] = (0, 2]$$

As $n \to \infty$, the left endpoint $\frac{1}{n} \to 0$, but 0 is never included. So the union is the half-open interval $(0, 2]$, which is not closed.

**Insight:** Arbitrary unions of closed sets need not be closed.

# Problem 6

**Q:** Prove that the (I)complement of an open set is closed, and that (ii) the complement of a closed set is open.

**(i) Proof:** Let $U \subseteq \mathbb{R}^n$ be open. $U^c$ is its complement, which is closed. Let $x$ be a limit point of $U^c$. Suppose, $x \notin U^c$ ie, $x \in U$. Since $U$ is open, $\exists N_\varepsilon(x) \subseteq U$.

If, $N_\varepsilon(x) \subseteq U$, then $N_\varepsilon(x)$ has no overlap with $U^c$. But, we assumed, $x$ was a limit point of $U^c$. And, we have a **contradiction!**

Thus, $x \notin U^c$ is false. Hence, for any limit point $x$ of $U^c$, $x \in U^c$.

$$\therefore U^c \quad \text{is closed}$$

# Problem 6

**(ii) Proof:** Let $C$ be closed. Then $C$ contains all its limit points. Consider its complement $U^c$. We must show: every point of $U^c$ is an interior point.

- If $x$ were not an interior point, then every neighborhood of $x$ would intersect $C$.
- That would make $x$ a limit point of $C$.
- Since $C$ is closed, all its limit points must lie in $C$.
- But $x \notin C$. We have a **Contradiction!**

Therefore, $x$ must be an interior point of $U^c$.

$$\therefore U^c \quad \text{is open}$$

# Thank you!

noyelaniyara@gmail.com