Microeconomics Tutorial II : Preference Relations

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Contents

Background

Let $\succeq$ be a weak preference relation on $X \subseteq \mathbb{R}^n_+$.

- $\succeq$ is assumed to be a complete pre-order (reflexive, transitive, complete).
- We derive a strict preference $\succ$ and indifference $\sim$ relation from $\succeq$.
- A strict preference relation $\succ$ is defined as:

$$x \succ y \iff (x \succeq y \quad \land \quad \neg(y \succeq x)).$$

- To show that $\succ$ is a **partial order**, i.e.,
  1. Irreflexive
  2. Transitive
  3. Antisymmetric

Irreflexivity of $\succ$

Suppose $x \succ x$.

- By reflexivity of $\succeq$, we always have $x \succeq x$.
- If $x \succ x$ were true, then we would also need $\neg(x \succeq x)$.
- This is a direct contradiction. Hence $\neg(x \succ x)$

$$\therefore \succ \text{ is irreflexive.}$$

Transitivity of $\succ$

Claim: If $x \succ y$ and $y \succ z$, then $x \succ z$.

$$\text{We need to show } x \succ z \implies x \succeq z \land \neg(z \succeq x)$$

Proof:

- $x \succ y \implies x \succeq y$ and $\neg(y \succeq x)$.
- $y \succ z \implies y \succeq z$ and $\neg(z \succeq y)$.

Transitivity of $\succ$

Claim: If $x \succ y$ and $y \succ z$, then $x \succ z$.

$$\text{We need to show } x \succ z \implies x \succeq z \land \neg(z \succeq x)$$

Proof:

- $x \succ y \implies x \succeq y$ and $\neg(y \succeq x)$.
- $y \succ z \implies y \succeq z$ and $\neg(z \succeq y)$.
- From transitivity of $\succeq$: $x \succeq y$ and $y \succeq z \implies x \succeq z$.

Transitivity of $\succ$

Claim: If $x \succ y$ and $y \succ z$, then $x \succ z$.

$$\text{We need to show } x \succ z \implies x \succeq z \wedge \neg(z \succeq x)$$

Proof:

- $x \succ y \implies x \succeq y$ and $\neg(y \succeq x)$.
- $y \succ z \implies y \succeq z$ and $\neg(z \succeq y)$.
- From transitivity of $\succeq$: $x \succeq y$ and $y \succeq z \implies x \succeq z$.
- Now, to show $\neg(z \succeq x)$; say $z \succeq x$. Then by transitivity, $z \succeq y$.
- But this contradicts $\neg(z \succeq y)$.

$$\therefore\ x \succ z.$$

Antisymmetry of $\succ$

Claim: If $x \succ y$ and $y \succ x$, then $x = y$.

Proof:

- Suppose $x \succ y$. Then $x \succeq y$ and $\neg(y \succeq x)$.
- Suppose also $y \succ x$. Then $y \succeq x$ and $\neg(x \succeq y)$.
- These two cannot hold simultaneously.
- Thus, the situation $x \succ y$ and $y \succ x$ never occurs unless $x = y$.

$$\therefore \succ \text{ is antisymmetric (vacuously true).}$$

Indifference Relation

The **indifference relation** $\sim$ is defined from the weak preference relation $\succeq$ as:

$$\forall \quad x, y \in X, x \sim y \Leftrightarrow (x \succeq y \text{ and } y \succeq x).$$

x and y are indifferent if each is weakly preferred to the other.

To prove $\sim$ is an equivalence relation, we need to show it's

1. Reflexive: $x \sim x$ for all x
2. Transitive: $x \sim y$ and $y \sim z$, then $x \sim z$
3. Symmetric: $x \sim y$ then $y \sim z$

Reflexivity of $\sim$

Claim: $\sim$ is reflexive.
For any $x \in X$:

$$x \succeq x \ \wedge x \succeq x \quad \text{(since } \succeq \text{ is reflexive).}$$

Thus:

$$x \sim x.$$

Transitivity of $\sim$

Claim: $\sim$ is transitive.
Suppose $x \sim y$ and $y \sim z$.

$$x \sim y \quad \Rightarrow \quad x \succeq y, \; y \succeq x$$

$$y \sim z \quad \Rightarrow \quad y \succeq z, \; z \succeq y$$

From transitivity of $\succeq$:

$$x \succeq y, \; y \succeq z \quad \Rightarrow \quad x \succeq z$$

$$z \succeq y, \; y \succeq x \quad \Rightarrow \quad z \succeq x$$

Hence:

$$x \succeq z \text{ and } z \succeq x \quad \Rightarrow \quad x \sim z.$$

Symmetry of $\sim$

Claim: $\sim$ is symmetric.
Suppose $x, y \in X$ such that $x \sim y$.

$$x \sim y \;\; \Rightarrow \;\; x \succeq y \text{ and } y \succeq x$$

This condition also implies:

$$y \succeq x \text{ and } x \succeq y$$

Therefore:

$$y \sim x. \quad \text{ie, } \sim \text{ is symmetric}$$

Indifference Set

iii:Q3: $\sim$ partitions X into indifference classes $\sim(x) := \{\, y \in X \mid x \sim y \,\}$ for all $x \in X$

Let $\succeq$ be a weak preference on X (reflexive, transitive, complete).

$$x \sim y \quad :\Longleftrightarrow \quad (x \succeq y \;\wedge\; y \succeq x)$$

For each $x \in X$, define the *indifference class*

$$\sim(x) := \{\, y \in X \mid x \sim y \,\}, \qquad \mathcal{I} := \{\sim(x) \mid x \in X\}.$$

Goal: show $\mathcal{I}$ is a *partition* of X.

What "partition" means (three checkpoints)

To prove $\mathcal{I}$ is a partition of X, we show:

1. **Nonemptiness:** $\forall x \in X, \ \sim(x) \neq \varnothing$ and $x \in \sim(x)$.
2. **Equal-or-disjoint:** For any $x, y \in X$, either $\sim(x) = \sim(y)$ or $\sim(x) \cap \sim(y) = \varnothing$.
3. **Covering:** $\bigcup_{x \in X} \sim(x) = X$.

Proof 1: Nonempty & $x \in\sim(x)$

Claim. For every $x \in X$, $\sim(x) \neq \varnothing$ and $x \in\sim(x)$.
Step 1. Reflexivity of $\succeq$ gives $x \succeq x$.
Step 2. By definition of $\sim$, from $x \succeq x$ and $x \succeq x$ we get $x \sim x$.
Step 3. Because $x \sim x$, the element x satisfies the membership condition of $\sim(x)$, hence $x \in\sim(x)$.
Conclusion. $\sim(x)$ contains at least x, so it is nonempty.

Proof 2: Equal or Disjoint

Case A: $\sim(x) = \sim(y)$.

1. Let $a \in \sim(x) \cap \sim(y)$. Then $x \sim a$ and $y \sim a$.
2. Symmetry of $\sim$ gives $a \sim x$ and $a \sim y$.
3. Transitivity of $\sim$: from $x \sim a$ and $a \sim y$ conclude $x \sim y$.
4. Inclusions:
   - If $b \in \sim(x)$ then $x \sim b$. Since $y \sim x$ and $x \sim b$, transitivity gives $y \sim b$, so $b \in \sim(y)$. Hence $\sim(x) \subseteq \sim(y)$.
   - By symmetry of the argument, $\sim(y) \subseteq \sim(x)$.

Hence,

$$\sim(x) = \sim(y)$$

Proof 2

Case B: $\sim(x) \cap \sim(y) = \varnothing$. We need to show $\neg(x \sim y)$

Suppose $\sim(x) \cap \sim(y) \neq \varnothing$

Let $z \in \sim(x) \cap \sim(y)$

Then $x \sim z$ and $y \sim z$. By symmetry of $\sim \Rightarrow z \sim y$

$\therefore$ we have $x \sim z \wedge z \sim y \Rightarrow x \sim y$ (transitivity)

Proof 2

Case B: $\sim(x) \cap \sim(y) = \varnothing$. We need to show $\neg(x \sim y)$

Suppose $\sim(x) \cap \sim(y) \neq \varnothing$

Let $z \in \sim(x) \cap \sim(y)$

Then $x \sim z$ and $y \sim z$. By symmetry of $\sim \Rightarrow z \sim y$

$\therefore$ we have $x \sim z \wedge z \sim y \Rightarrow x \sim y$ (transitivity)

This contradicts the case $\neg(x \sim y)$.

Hence, $\sim(x) \cap \sim(y) \neq \varnothing$ is a false

Proof 3: Covering X

Claim. $\bigcup_{x \in X} \sim(x) = X$.

1. Let $x \in X$ be arbitrary.
2. By Proof 1, $x \sim x$, hence $x \in \sim(x)$.
3. Since x was arbitrary, every element of X lies in at least one class $\sim(x)$.

Conclusion. The union of all classes equals X.

Lexicographic Preference

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $X \subseteq \mathbb{R}^2$.

$$x \succeq^L y \iff [x_1 > y_1] \text{ or } [x_1 = y_1 \text{ and } x_2 \geq y_2].$$

- Notation:
  - $x \succeq^L y$: "x is at least as good as y (lexicographically)".
  - $x \succ^L y$: strict lexicographic preference.
  - $x \sim^L y$: indifference (here $x \sim^L y \iff x = y$).
- Q2: Show that the lexicographic preference ordering is a complete pre-ordering and also antisymmetric.
  1. **Complete preorder**: reflexive, transitive, and complete (comparability), and
  2. **Antisymmetric**: if $x \succeq^L y$ and $y \succeq^L x$ then $x = y$.

A. Reflexivity

Claim. $\forall x \in X,\ x \succeq^L x.$

Proof:

$$x_1 = x_1 \quad \text{and} \quad x_2 \geq x_2,$$

so the second clause of the definition holds and $x \succeq^L x$.

Any bundle is at least as good as itself.

$\therefore$ Lexicographic preferences are reflexive

B. Completeness (comparability)

Claim. For any $x, y \in X$, either $x \succeq^L y$ or $y \succeq^L x$.

Proof:

- If $x_1 > y_1$, then $x \succeq^L y$.
- If $x_1 < y_1$, then $y \succeq^L x$.
- If $x_1 = y_1$, then compare second coordinates:
    - If $x_2 \geq y_2$ then $x \succeq^L y$.
    - If $y_2 \geq x_2$ then $y \succeq^L x$.

$$\therefore x, y \in X, \text{ we have } x \succeq^L y \text{ or } y \succeq^L x.$$

C. Transitivity

Claim. If $x \succeq^L y$ and $y \succeq^L z$ then $x \succeq^L z$.

Proof: From $x \succeq^L y$ we have either

$$(i)\ x_1 > y_1 \quad \text{or} \quad (ii)\ x_1 = y_1 \wedge x_2 \geq y_2.$$

From $y \succeq^L z$ we have either

$$(a)\ y_1 > z_1 \quad \text{or} \quad (b)\ y_1 = z_1 \wedge y_2 \geq z_2.$$

Consider the four combinations:

1. (i) and (a): then $x_1 > y_1 > z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.

C. Transitivity

Claim. If $x \succeq^L y$ and $y \succeq^L z$ then $x \succeq^L z$.

Proof: From $x \succeq^L y$ we have either

$$(i)\ x_1 > y_1 \quad \text{or} \quad (ii)\ x_1 = y_1 \wedge x_2 \geq y_2.$$

From $y \succeq^L z$ we have either

$$(a)\ y_1 > z_1 \quad \text{or} \quad (b)\ y_1 = z_1 \wedge y_2 \geq z_2.$$

Consider the four combinations:

1. (i) and (a): then $x_1 > y_1 > z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.
2. (i) and (b): then $x_1 > y_1 = z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.

C. Transitivity

Claim. If $x \succeq^L y$ and $y \succeq^L z$ then $x \succeq^L z$.

Proof: From $x \succeq^L y$ we have either

$$(i)\ x_1 > y_1 \quad \text{or} \quad (ii)\ x_1 = y_1 \wedge x_2 \geq y_2.$$

From $y \succeq^L z$ we have either

$$(a)\ y_1 > z_1 \quad \text{or} \quad (b)\ y_1 = z_1 \wedge y_2 \geq z_2.$$

Consider the four combinations:

1. (i) and (a): then $x_1 > y_1 > z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.
2. (i) and (b): then $x_1 > y_1 = z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.
3. (ii) and (a): then $x_1 = y_1 > z_1$, so $x_1 > z_1 \Rightarrow x \succeq^L z$.
4. (ii) and (b): then $x_1 = y_1 = z_1$ and $x_2 \geq y_2 \geq z_2$, hence $x_2 \geq z_2$ and $x \succeq^L z$.

Thus in every case $x \succeq^L z$. Therefore $\succeq^L$ is transitive.

D. Antisymmetry

Claim. If $x \succeq^L y$ and $y \succeq^L x$, then $x = y$.

Proof (by contradiction). Assume $x \succeq^L y$ and $y \succeq^L x$ but $x \neq y$.

- If $x_1 \neq y_1$, then either $x_1 > y_1$ or $y_1 > x_1$.
  - If $x_1 > y_1$, then $x \succeq^L y$ holds by the first clause, but $y \succeq^L x$ cannot hold because $y_1 < x_1$. Contradiction.
  - Similarly if $y_1 > x_1$.
- Hence $x_1 = y_1$. If also $x_2 \neq y_2$, then from $x \succeq^L y$ we deduce $x_2 > y_2$, while $y \succeq^L x$ gives $y_2 > x_2$. Contradiction.

Therefore no contradiction-free alternative exists; so $x = y$.

Thank you!

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