Microeconomics Tutorial III : Convexity & Marshallian Demand

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Contents

Marshallian Demand and Convexity

Convex Preferences $\iff$ Convex Upper-Contour Sets

Claim. A preference relation $\succeq$ on $X \subseteq \mathbb{R}^n_+$ is convex iff for every $x \in X$ the set

$$\succeq(x) := \{y \in X \mid y \succeq x\}$$

is convex.

Proof.
(a) Convex preferences $\Rightarrow$ each $\succeq(x)$ convex.
Say, $x \in X$. Take any $y, z \in \succeq(x)$. By definition $y \succeq x$ and $z \succeq x$.

Since $\succeq$ is convex, for every $\alpha \in [0, 1]$

$$\alpha y + (1 - \alpha)z \succeq x.$$

Hence $\alpha y + (1 - \alpha)z \in \succeq(x)$. Therefore $\succeq(x)$ is convex.

Convex Preferences $\iff$ Convex Upper-Contour Sets

(b) Each $\succeq (x)$ convex $\Rightarrow$ convex preferences. Assume every $\succeq (x)$ is convex. Take any $x \in X$ and suppose $y \succeq x$ and $z \succeq x$.

Then $y, z \in \succeq (x)$ and convexity of $\succeq (x)$ implies:

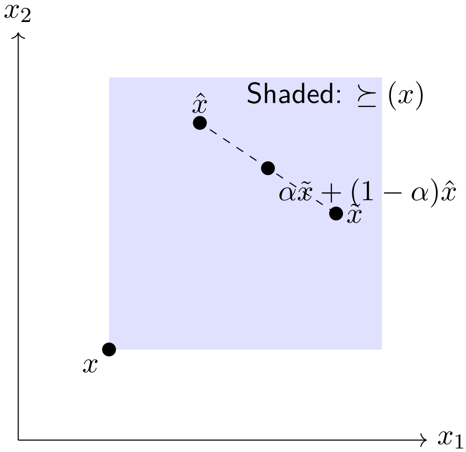
$$\alpha y + (1-\alpha) z \in \succeq (x) \quad \forall \alpha \in [0,1]$$

i.e.

$$\alpha y + (1-\alpha) z \succeq x$$

Thus $\succeq$ is convex.

Convex Upper Contour Sets: Geometric Intuition



Recap

- **Budget Set:** $B(p, y) = \{x \in X \mid p \cdot x \leq y\}$
- **Demand Set:** $\Phi(p, y) = \{x \in B(p, y) \mid x \succeq z \ \ \forall z \in B(p, y)\}$
- **Convex Preferences:** If $\tilde{x} \succeq x$ and $\hat{x} \succeq x$, then $\alpha\tilde{x} + (1 - \alpha)\hat{x} \succeq x$

Convexity of the Marshallian Demand Set

Theorem. If the preference relation $\succeq$ is convex, then the Marshallian demand set $\Phi(p, y)$ is convex.

Proof.

- Let $\tilde{x}, \hat{x} \in \Phi(p, y)$. By definition:

$$\tilde{x}, \hat{x} \in B(p, y), \quad \tilde{x} \succeq z,\ \hat{x} \succeq z \quad \forall z \in B(p, y).$$

- Take a convex combination:

$$x' = \alpha\tilde{x} + (1 - \alpha)\hat{x}, \quad \alpha \in [0, 1].$$

- Since $B(p, y)$ is convex:

$$p \cdot x' = \alpha p \cdot \tilde{x} + (1 - \alpha)p \cdot \hat{x} \leq y,$$

  so $x' \in B(p, y)$. By convexity of preferences:

$$x' \succeq z \quad \forall z \in B(p, y).$$

Hence $x' \in \Phi(p, y)$. $\therefore \Phi(p, y)$ is convex

Strict Convexity $\implies$ Singleton Demand

Recap.

- **Marshallian Demand Set:**

$$\Phi(p, y) = \{x \in B(p, y) \mid x \succeq z \;\; \forall z \in B(p, y)\}$$

- **Convex Preferences:** Mixtures are at least as good as extremes:

$$\alpha \tilde{x} + (1 - \alpha)\hat{x} \succeq x$$

- **Strict Convexity:** If $\tilde{x} \sim \hat{x}$ with $\tilde{x} \neq \hat{x}$, then

$$\alpha \tilde{x} + (1 - \alpha)\hat{x} \succ \tilde{x}, \hat{x} \quad \forall \alpha \in (0, 1).$$

Mixes strictly better than extremes unless bundles are identical.

Proof: Strict Convexity $\implies$ Singleton Demand

Claim: If $\succeq$ is strictly convex, then $\Phi(p, y)$ is a singleton.

- Suppose not: let $\tilde{x}, \hat{x} \in \Phi(p, y)$ with $\tilde{x} \neq \hat{x}$. Then both are optimal, so $\tilde{x} \sim \hat{x}$.
- By strict convexity, for any $\alpha \in (0, 1)$:

$$x' = \alpha\tilde{x} + (1 - \alpha)\hat{x} \succ \tilde{x}, \hat{x}.$$

- Since $p \cdot x' = \alpha p \cdot \tilde{x} + (1 - \alpha)p \cdot \hat{x} \leq y$, we have $x' \in B(p, y)$ and $x' \succ \tilde{x}, \hat{x}$.
- But then $\tilde{x}, \hat{x}$ cannot be optimal — contradiction.

$$\implies \Phi(p, y) = \{x^*\}, \quad \text{a singleton.}$$

Thank you!

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